A MODEL FOR RESIDENCE TIME IN CONCURRENT VARIABLE INTERVAL PERFORMANCE

MICHAEL A. NAVAKATIKYAN

UNIVERSITY OF AUCKLAND AND
BRAINZ INSTRUMENTS LTD, AUCKLAND, NEW ZEALAND

A component-functions model of choice behavior is proposed for performance on interdependent concurrent variable-interval (VI) variable-interval schedules based on the product of two component functions, one that enhances behavior and one that reduces behavior. The model is the solution to the symmetrical pair of differential equations describing behavioral changes with respect to two categories of reinforcers: enhancing and reducing, or excitatory and inhibitory. The model describes residence time in interdependent concurrent VI VI schedules constructed from arithmetic and exponential distributions. The model describes the data reported by Alsop and Elliffe (1988) and Elliffe and Alsop (1996) with a variance accounted for of 87% compared to 64% accounted for by the Davison and Hunter (1976) model and 42% by Herrnstein's (1970) hyperbola. The model can explain matching, undermatching, and overmatching in the same subject under different procedures and has the potential to be extended to performance on concurrent schedules with more than two alternatives, multiple schedules, and single schedules. Thus, it can be considered as an alternative to Herrnstein's quantitative law of effect.

Key words: choice, concurrent schedules, VI schedules, interdependent schedules, mathematical models, matching, law of effect, pigeons

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Herrnstein (1970) suggested that there was a direct proportionality between relative response rate and relative reinforcer rate. This general principle allowed Herrnstein to propose a hyperbolic function for the response–reinforcer relation in single schedules, a further model for multiple-schedule performance, and, in particular, a further model for performance in concurrent schedules. As

more data became available, the generalized matching law (Baum, 1974) replaced Herrnstein's matching law as a tool for the description of the relationship between response and reinforcer ratios. Davison and Hunter's (1976) generalized matching equation was offered as a molar model for the absolute values of behavioral measures in single and multiple schedules. Yet there were no published attempts to assess it, though appropriate concurrent variable-interval variable-interval (conc VI VI) schedule data are available, especially a large set of data reported by Alsop and Elliffe (1988) and Elliffe and Alsop (1996). The goal here is to explore molar models of choice behavior, in particular for interdependent conc VI VI schedules, and to suggest an alternative principle and a model of choice that can also be applied to single-schedule and multiple-schedule performances.

The Matching Law

The equality between relative frequency of responding and relative frequency of reinforcement obtained in concurrent VI VI procedures was called the matching law (Herrnstein, 1961). The related general principle (and a molar model) of behavior stated that the absolute rate of responding on an

alternative in a choice procedure is proportional to its associated relative reinforcer rate (Herrnstein, 1970):

$$B_1 = \frac{kR_1}{\sum\limits_{i=0}^{n} R_i}. \tag{1}$$

In Equation 1, B is the absolute rate of responding (responses per min), R is the absolute rate of reinforcement (reinforcers per min), k is a constant, and i is an index that covers all alternative responses. Matching takes place over any set of alternative responses for which k and ΣR_i are fixed (Herrnstein, 1970). The constancy of k represents ''the total amount of behavior generated by all the reinforcements operating on the subject at a given time'' (Herrnstein, 1974, p. 161), that is,

$$k = \sum\limits_{i=0}^{n} B_i. \tag{2}$$

Consequently, for single schedules of reinforcement, the response–reinforcer rate function is a hyperbola, where k and R_o are constants, with R_o the unknown aggregate reinforcer for other responses:

$$B = kR/(R + R_0). \tag{3}$$

For concurrent schedules, the total rate of reinforcement (the denominator in Equation 1) is the same for each alternative, so the matching relation between relative response rates and relative reinforcer rates is expressed as:

$$B_1 = kR_1/(R_1 + R_2 + R_0); \tag{4}$$

thus

$$B_1/(B_1 + B_2) = R_1/(R_1 + R_2), \tag{5}$$

where subscripts 1 and 2 denote the first and the second alternatives. Equation 4 also describes a contrast effect—an increase in response rate in one component when the reinforcer rate in the other component is decreased.

The equation for multiple-schedule performance (Herrnstein, 1970) required the introduction of a new constant m:

$$B_1 = kR_1/(R_1 + m \cdot R_2 + R_0). \tag{6}$$

Herrnstein assumed that the value of m depended on the degree of interaction between components; for concurrent schedules it is 1 (maximal interaction), but for multiple schedules the interaction is weaker and m is less than 1. Equation 6 also predicts, at least ordinally, the contrast effect observed in concurrent and multiple schedules (Herrnstein, 1970; Reynolds, 1963; Williams, 1983).

Generalized Matching

Baum (1974) showed systematic deviations from strict matching between relative response and relative reinforcer frequencies and proposed a power function alternative, the generalized matching law:

$$\frac{B_1}{B_2} = c \cdot (\frac{R_1}{R_2})^a, \tag{7}$$

that, in double logarithmic coordinates, is a linear function:

$$\log(\frac{B_1}{B_2}) = a \cdot \log(\frac{R_1}{R_2}) + \log c. \tag{8}$$

Deviations from strict matching are accommodated by two parameters, called bias (c) (Baum, 1974) and sensitivity to reinforcement (a) (Lander & Irwin, 1968; Lobb & Davison, 1975). The latter is more accurately termed sensitivity to log reinforcer ratios (Davison & McCarthy, 1994). A value of sensitivity less than 1.0, called undermatching, is now considered the normative finding in choice behavior on conc VI VI schedules (Baum, 1979; Davison & McCarthy, 1988; Taylor & Davison, 1983; Wearden & Burgess, 1982; Williams, 1988), though an increase in the changeover requirement shifts performance toward overmatching (Pliskoff & Fetterman, 1981). The use of double logarithmic coordinates and a power function was demonstrated by Baum and Rachlin (1969), Hollard and Davison (1971), Lander and Irwin (1968), Staddon, (1968), and White and Davison (1973).

The generalized matching law fitted the data very well, though more recent findings have suggested that the sensitivity parameter is not invariant under different overall reinforcer rates. Using arithmetic VI schedules, Alsop and Elliffe (1988), Fantino, Squires, Delbrück, and Peterson (1972), and Logue and Chavarro

(1987) reported that sensitivity increased with increases in overall reinforcer rates. Elliffe and Alsop (1996), using exponential schedules, reported a similar increase in sensitivity with increases in overall reinforcer rates followed by a decrease at high overall reinforcer rates. Also, both undermatching and overmatching were found in the same subject's performance when overall reinforcer rates were varied.

Numerous approaches have been developed to explain matching or generalized matching by reference to other basic principles of behavior (see Baum & Aparicio, 1999; Davison & Baum, 2000; Davison & McCarthy, 1988; MacDonall, 1999, 2000; Nevin, 1984; Shimp, 1966, 1969; Williams, 1988), but I concentrate here on molar models for the absolute (as distinguished from relative, or ratio) values of behavioral measures. Thus I omit, for example, the generalized matching law per se (Equations 7 and 8), the momentum approach of Nevin and his associates (Nevin, 1974; Nevin, Mandell & Atak, 1983; Nevin & Grace, 2000) and the ''pressure'' model for matching and contrast suggested by Staddon (1983). The justification is that an accurate model for absolute values of behavioral measures should accurately predict ratio values, whereas the reverse is not possible.

Existing Molar Models for the Absolute Values of Behavioral Parameters

Baum (1974, 1979) pointed out that Equation 7 is related to Stevens's (1957) psychophysical law:

$$S = cI^a, \qquad (9)$$

where c and a are constants, sensation (S) is the analog of response rate, and the stimulus intensity (I) is the analog of reinforcer rate. If response rate for each alternative is a power function, then the ratio of response rates is Equation 8 with $log\ c = 0$. (Note also that the constant c has different meanings in Equations 9 and 8.) Yet, as pointed out by Herrnstein (1970), no power law (which is an infinitely rising function) can accommodate response-rate data from single-key procedures.

Davison and Hunter (1976) suggested a range of other molar models, called generalized matching equations, that include absolute response rate and transform into an unbiased form of Equation 7:

$$B_1 = k'(\frac{R_1}{\sum R_i})^a; \qquad (10)$$

or, alternatively,

$$B_1 = k' \frac{R_1^a}{\sum (R_i{}^a)}, \qquad (11)$$

where k' and a are constants, k' is an analog to Herrnstein's k, and a is an analog to the generalized matching sensitivity parameter. (R_0—the aggregated other reinforcer rate—is included in the reinforcer sum in the denominator.)

McDowell (1986) suggested a ''weak form'' of matching for performance in concurrent schedules, with each pair of reinforcers having its own sensitivity and bias, thus using 8 degrees of freedom for a two-alternative concurrent schedule model that includes extraneous reinforcer rate:

$$B_1 = k'\left[\frac{1}{b_{10}}\left(\frac{R_0}{R_1}\right)^{a10} + \frac{1}{b_{12}}\left(\frac{R_2}{R_1}\right)^{a12} + 1\right]^{-1} \qquad (12)$$

and

$$B_2 = k'\left[\frac{1}{b_{20}}\left(\frac{R_e}{R_2}\right)^{a20} + b_{12}\left(\frac{R_1}{R_2}\right)^{a12} + 1\right]^{-1}, \qquad (13)$$

where the parameters a and b are sensitivities and biases for the generalized matching law between pairwise combinations of the three alternatives (the first, second, and zero, or aggregated other reinforcers).

Furthermore, Killeen (1982) suggested a molar application of his incentive theory for response rates in concurrent schedules, but it is not considered here.

The goal of the present paper is to identify a principle and to develop a model for choice behavior that is applicable to conc VI VI schedule performance, that is, a model that accurately describes residence time data, and allows for matching, overmatching, and undermatching in the performance of the same subject. For the purposes of assessing the models that will be considered, I used the extensive data reported by Alsop and Elliffe (1988) and Elliffe and Alsop (1996).

ACCURACY OF EXISTING MOLAR MODELS

The Experimental Data and Procedure

There are two types of conc VI VI schedules: arithmetic and exponential. These differ in the progression used to generate the intervals comprising the VI schedule. A typical arithmetic VI schedule comprises a series of discrete intervals that are randomized from the first n terms of the progression x, $3x$, $5x$, $7x$, … , where x = mean interval/n. The probability of arranging the next reinforcer under arithmetic schedules increases with time and reaches 1 at the maximum scheduled time interval. A typical exponential VI schedule interrogates a probability generator every 1 s and arranges a reinforcer with probability = 1/mean interval. Under exponential schedules, the probability of arranging the next reinforcer is constant over time, and the time between pairs of reinforcers can theoretically be infinite.

Concurrent schedules also can be arranged in two different ways with regard to their interaction. With independent scheduling, arranging a reinforcer for one response does not affect the availability of a reinforcer for another response. In interdependent concurrent schedules (Stubbs & Pliskoff, 1969), when one schedule arranges a reinforcer, the other schedule(s) stop timing until that reinforcer is collected. The data from the series of interdependent concurrent arithmetic (Alsop & Elliffe, 1988) and exponential VI schedules (Elliffe & Alsop, 1996) mentioned above were obtained using the same 6 pigeons and provide the basis for the assessment of existing models and of the new model proposed here. Pigeons were exposed to a broad range of overall and relative reinforcer rates. Programmed overall reinforcer rates for the arithmetic schedules were 0.22, 0.44, 1, 2, 5, and 10 reinforcers per min, and for the exponential schedules they were 0.25, 0.50, 1, 2, 5, and 10 reinforcers per min. Five or six different relative reinforcer rates were arranged at each overall reinforcer rate, usually with values of 0.1, 0.2, 0.5, 0.8, and 0.9. The changeover delay was 2 s. For the purpose of modeling, the data from each pigeon and each type of VI schedule (arithmetic or exponential) were used, forming 12 data sets. Thus, 12 fits (6 pigeons × 2 schedule types) were calculated to describe the effect of various reinforcer rates on residence time for each molar model that was assessed. Residence time is the uninterrupted time spent responding on each key, measured from the first response on a key until the next response on the other key. It excludes the time occupied by reinforcer delivery. Residence times were calculated from total time spent responding on each key divided by half of the total changeovers over the last five sessions in each condition for each subject (see Appendices of Alsop & Elliffe, 1988; Elliffe & Alsop, 1996).

Model Selection and Statistics

The model parameters were optimized to obtain the minimum sum of squared deviations of the estimated data values from the actual data values using the Corel QuatroPro8 optimizer. The best fits were obtained by optimizing log residence times rather than residence times per se, thus decreasing the influence of extreme values of residence times on the fits. The differences between parameter values for the arithmetic and exponential models were analyzed using a paired two-tailed *t*-test.

Because the models assessed here have different numbers of adjustable parameters, it is not enough to calculate variance accounted for (VAC) in order to select a better model. Two approaches are used here that include terms that penalize additional degrees of freedom found in models: the Akaike information criterion (AIC) and the Bayesian information criterion (BIC), with BIC being the more conservative criterion (i.e., providing a larger penalty for the range of sample sizes and the number of adjustable parameters that are used). AIC is recommended in preference to BIC for biological modeling (Burnham & Anderson, 1998). In what follows, both were used to demonstrate the robustness of the conclusions.

When the sample size is small (i.e., the value of the ratio n/K is less than 40), the second-order Akaike information criterion is recommended (Burnham & Anderson, 1998):

$$AICc = n \cdot \log_e \frac{RSS}{n} + 2K\left(\frac{n}{n-K-1}\right), \quad (14)$$

where $AICc$ is the second-order Akaike information criterion in units of information, n is number of data points used, K is number of

adjustable parameters in a model plus 1, and *RSS* is the residual sum of squares for the fitted model. Better models have smaller values of AICc, which describes the information lost in replacing actual results with the model's results. The last term in the expression is the "penalty" term, and it increases as the number of adjustable parameters increases.

The Bayesian information criterion for a data set differs by the penalty term and is calculated as:

$$BIC = n \cdot \log_e \frac{RSS}{n} + K \log_e (n). \qquad (15)$$

When AICc and BIC are calculated for a series of data sets (i.e., 12 fits for the models considered here), the information criteria are calculated by Equations 16 and 17 (B. McArdle, personal communication, 2005):

$$AICc = \sum_{i=1}^{N} \left(n_i \cdot \log_e \frac{RSS}{n_i} \right) \\ + 2KN \left(\frac{n_t}{n_t - KN - N} \right) \text{ and} \qquad (16)$$

$$BIC = \sum_{i=1}^{N} \left(n_i \cdot \log_e \frac{RSS}{n_i} \right) + KN \log_e (n_t), \qquad (17)$$

where N is number of data sets ($N = 12$), i is index of a data set 1, 2, … N, and n_t is the total number of data points used in the set (i.e., in all N datasets).

Values of AICc and BIC by themselves have no meaning for a given data set, as they depend on the dimension of *RSS*, that is, the residence time measured in seconds will produce an *RSS* value 3600 (60^2) times larger than if it is expressed in minutes. Thus, models were compared using AICc and BIC differences that do not depend on the dimension of *RSS*:

$$\Delta_i = AICc_i - AICc_{\min} \text{ and} \qquad (18)$$

$$\Delta_i = BIC_i - BIC_{\min}, \qquad (19)$$

where $AICc_i$ and BIC_i are the Akaike and Bayesian criteria for the ith model, and $AICc_{min}$ and BIC_{min} are Akaike and Bayesian criteria for the best model (or the minimum AICc and BIC value). The level of support for a model depends on the value of Δ_i. The relative likelihood (RL_i), which is the probability that

a particular i^{th} model is a better model than the model with the minimum value of an information criterion, is given by:

$$RL_i = \exp\left(-\frac{1}{2}\Delta_i\right). \qquad (20)$$

The information criteria must be calculated using the same data set for each model. After testing, the models are ranked in order of increasing values of AICc and BIC, and the set of models within a cutoff value of AICc and BIC differences from the best model is considered as a confidence set of models, each of them having some chance to be the true best model. A cutoff value of $\Delta_i = 6$ is recommended, as it provides a relative likelihood value of less than 0.05 (as $exp(-6/2) = .0498$). A cutoff difference of 10 or more means that there is virtually no support for a model being a better model than the model with the minimum value of the information criterion (Burnham & Anderson, 1998).

Models of Residence Time Based on Existing Molar Approaches

Five residence-time models for the data of Alsop and Elliffe (1988) and Elliffe and Alsop (1996) were derived using Stevens' law (Stevens, 1957), Herrnstein's hyperbola (Herrnstein, 1970), two models developed by Davison and Hunter (1976), and one developed by McDowell (1986). The equations for all five residence-time models are given in the Appendix A (Equations A1 to A10), but some are also presented here.

Values of the AICc and BIC, and the means of the VAC of the 12 fits (6 pigeons × 2 schedules) for each model are presented in Table 1. Both information criteria (AICc and BIC) were smallest for the Davison and Hunter (1976) Model 2 (Equations 21 and 22):

$$T_1 = k'(bR_1)^a / [(bR_1)^a + R_2^a + R_0^a] \text{ and} \qquad (21)$$

$$T_2 = k' R_2^a / [(bR_1)^a + R_2^a + R_0^a], \qquad (22)$$

where T_1 and T_2 are residence times on the first and second alternatives, R_1, R_2, and R_0 are obtained reinforcer rates on the first, second, and aggregated other alternatives, k' is an analog to "the total amount of behavior," a is an analog to the generalized matching sensitivity parameter, and b is a reinforcer bias. The difference from the next best model (Davison

Table 1

Accuracy of five models of residence time (see Appendix A, Equations A1 to A10) measured in terms of the second-order Akaike information criterion (AICc, Equation 16) and Bayesian information criterion (BIC, Equation 17) and as the percentage of variance accounted for (VAC). *df* is the number of adjustable parameters in a model, Δ_i is difference of a model information criterion from the best model (Equations 18 and 19). Models are ordered by the value of AICc and ranked separately by values of the AICc and BIC, with the best model ranked 1. SEM is the standard error, and VAC are averages for the 12 fits (6 pigeons $\times$ 2 schedule types) performed on the data of Alsop and Elliffe (1988) and Elliffe and Alsop (1996).

		AICc			BIC			VAC	
Model	*df*	Value	Δ_i	Rank	Value	Δ_i	Rank	Mean	SEM
Davison-Hunter 2	4	−2429	0	1	−2216	0	1	64	2
Davison-Hunter 1	4	−2348	81	2	−2135	81	2	60	2
McDowell	8	−2309	120	3	−1898	318	4	64	2
Herrnstein	3	−2107	322	4	−1946	270	3	42	4
Stevens	3	−1772	657	5	−1611	605	5	12	2

& Hunter Model 1) was 81 for both AICc and BIC (see Table 1), and the relative likelihood calculated by Equation 20, indicating that the Davison and Hunter Model 1 is the better model than the Davison and Hunter Model 2, was 2×10^{-18}. Thus, a comparison of the models suggests that the Davison and Hunter Model 2 is the best of the molar models that were tested, although its mean VAC (64%) was not large. The AICc and BIC differences of the Davison and Hunter Model 2 from Herrnstein's (1970) hyperbola-based model:

$$T_1 = kbR_1/(bR_1 + R_2 + R_0) \text{ and} \qquad (23)$$

$$T_2 = kR_2/(bR_1 + R_2 + R_0), \qquad (24)$$

where k is the ''total amount of behavior'', were even greater. The mean VAC for Herrnstein's hyperbola was only 42% on average (see Table 1).

The average parameter values of the Davison and Hunter (1976) Model 2 and Herrnstein's (1970) hyperbola for arithmetic and exponential schedules are presented in Table 2, with parameter values detailed for each pigeon and each model included in Appendix A (Table A1). For both models, the parameters related to ''total amount of behavior (k and k') were almost twice as large for the exponential schedule ($p < .001$ in both cases). The sensitivity parameter (a) of the Davison and Hunter Model 2 for exponential schedules

Table 2

Mean parameter values of Herrnstein's (1970) hyperbola (Appendix A, Equations A3 and A4) and the Davison and Hunter (1976) Model 2 (Equations 21 and 22). There are six fits of the models for arithmetic and exponential schedules, with 58 to 64 data points in each fit. k and k' are ''the total amount of behavior,'' R_o is the aggregated other reinforcer rate, a is an analog to the generalized matching sensitivity parameter, and b is a reinforcer bias. VAC is percentage of variance accounted for.

	Herrnstein's hyperbolic model				Davison & Hunter Model 2				
	Model parameters				Model parameters				
	k	R_o	log b	VAC	k'	R_o	a	log b	VAC
	Arithmetic schedule (Alsop & Elliffe, 1988)								
Mean ± SEM	27.3	0	−0.04	32	23.2	0	0.61	−0.07	60
	2.1	0	0.02	4	2.0	0	0.03	0.02	2
	Exponential schedule (Elliffe & Alsop, 1996)								
Mean ± SEM	46.7[+++]	0	−0.04	52	40.8[+++]	0	0.69[+]	−0.07	68
	4.6	0	0.02	5	4.3	0	0.03	0.05	3

[+] $p < .05$ for the comparison of means by 2-tailed paired *t*-test;   [+++] $p < .001$.

was also significantly ($p < .05$) larger than that for arithmetic schedules (Table 2). An interesting feature of both models, as well as all others containing R_0, was that, if this parameter was not constrained to be equal to or greater than 0, the value of R_0 was negative in all 12 fits of each model (see Appendix A, Table A1).

THE COMPONENT-FUNCTIONS MODEL FOR RESIDENCE TIME

The General Principle of Choice Behavior and the New Model for Residence Time

Herrnstein's (1970) model and the associated models described above are based on the matching (or the generalized matching) principle, which can be formulated as the distribution of a hypothetical entity—the total amount of behavior (Herrnstein, 1974)— according to some function of the fraction of total reinforcers allocated to a particular behavior.

Despite the apparent feasibility of behavior allocation according to relative reinforcers, the matching principle implies an inherent asymmetry in its treatment of different kinds of reinforcers, which are usually assigned to two categories. The first category consists of reinforcers produced by a particular response (or that the subject associates as such). The second category contains reinforcers from other sources (Catania, 1973). On Herrnstein's (1970) view, for example, these would be reinforcers available at the current alternative and reinforcers available at the other alternatives, including aggregated other reinforcers, respectively. According to the matching principle, both categories of reinforcers (which appear in the denominator of Herrnstein's hyperbola) have an inhibitory effect on responding, whereas the first category of reinforcers also exerts an excitatory effect.

As an alternative to this asymmetrical view, I hypothesized that each category of reinforcers (reinforcers produced by a response, and all other reinforcers) operates via its own independent function, which I term a *component function*. The combination of two such independent functions controls behavior. We term the function for the first category of reinforcers an *enhancing-component function* and the function for the second category

of reinforcers a *reducing-component function*. Accordingly, the first category of reinforcers may be referred to as *enhancing reinforcers*, and the second category of reinforcers as *reducing reinforcers*, meaning that they enhance and reduce a particular behavior, respectively. The enhancing-component function is an increasing function, and the reducing-component function is a decreasing function. I hypothesized further that, if the quantity of one of the reinforcers changes, the resulting change in behavior occurs in proportion to the change of its component function, and also in proportion to the other component function. (Here the mathematical synonym for ''change'' is the first derivative.) In accordance with this principle, a new behavioral steady state will be reached. The new state will be adaptive by virtue of the fact that it is adaptive to increase behavior if the rate of enhancing reinforcers has increased, and it is adaptive to decrease behavior if the rate of reducing reinforcers has increased.

The principle described above is mathematically a product of the component functions plus a constant:

$$T = F_{enh} \cdot F_{red} + T_a. \qquad (25)$$

where T is the resulting behavioral function, that is, residence time, F_{enh} and F_{red} are the enhancing- and reducing-component functions of enhancing and reducing reinforcers, respectively, and T_a is a constant.

Alternatively the principle can be expressed as a system of two partial differential equations:

$$\frac{\partial T}{\partial R_{enh}} = \frac{dF_{enh}}{dR_{enh}} \cdot F_{red} \text{ and} \qquad (26)$$

$$\frac{\partial T}{\partial R_{red}} = \frac{dF_{red}}{dR_{red}} \cdot F_{enh}, \qquad (27)$$

where R_{enh} and R_{red} are enhancing and reducing reinforcer rates; $\partial T/\partial R_{enh}$ and $\partial T/\partial R_{red}$ are partial derivatives (or changes) in T with a change in R_{enh} only and R_{red} only, respectively; dF_{enh}/dR_{enh} and dF_{red}/dR_{red} are full derivatives (changes) of component functions with a change in their respective reinforcers. The solution of this system is Equation 25, and Equations 26 and 27 warrant the appearance of T_a in Equation 25.
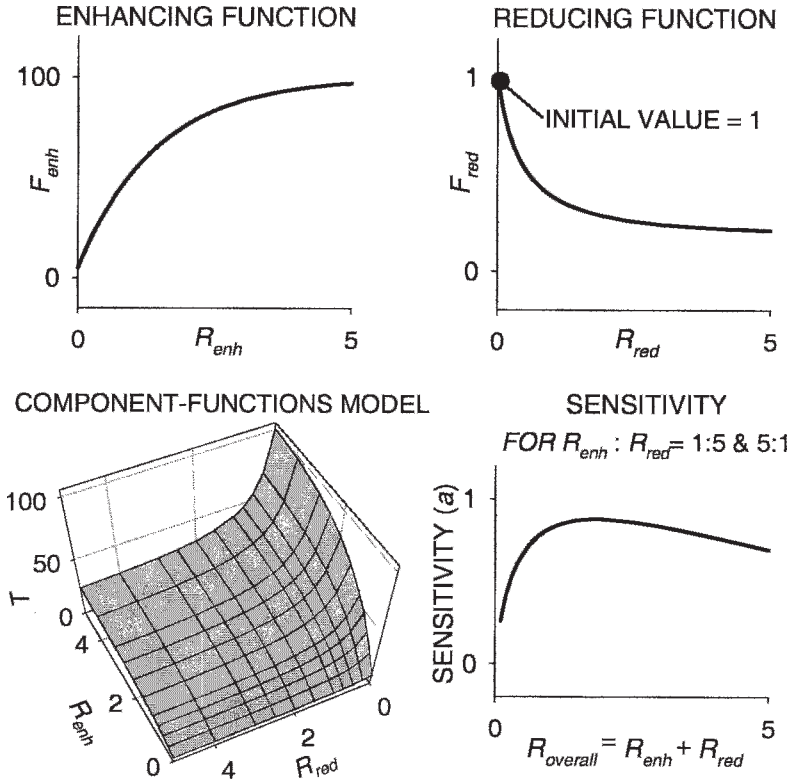
Fig. 1. A hypothetical model of behavior. Upper-left panel: The enhancing-component function (F_{enh}) of enhancing reinforcer rate (R_{enh}). Upper-right panel: The reducing-component function (F_{red}) of reducing reinforcer rate (R_{red}) with initial value = 1. Lower-left panel: Predicted residence time (T). Lower-right panel: generalized matching sensitivity values predicted from the model if calculated as a function of overall reinforcer rate with ratio $R_{enh} : R_{red} = 1 : 5$ and $5 : 1$ for each overall reinforcer rate that is, for T_1 the ratio $R_{enh} : R_{red} = R_1 : R_2$ (= 1 : 5); while for T_2 the ratio $R_{enh} : R_{red} = R_2 : R_1$ (= 5 : 1).

A hypothetical example of the prediction of residence time in two-alternative concurrent schedules using Equation 25 is shown in Figure 1. The enhancing-component function is an increasing function of the enhancing reinforcer rate (upper-left panel), and the reducing-component function is a decreasing function of the reducing reinforcer rate (upper-right panel). The value of the reducing-component function is chosen to be 1 when the reducing reinforcer rate equals 0, thus the decreasing-component function does not affect the resulting behavior in the absence of reducing reinforcers. The resulting behavior is displayed as a 3-D surface (lower-left panel). For residence time on the first alternative (T_1), the enhancing reinforcer rate is identified with R_1 and the reducing reinforcer rate with R_2 in Equation 28 below. For

residence time on the second alternative (T_2), the enhancing reinforcer rate is identified with R_2 and the reducing reinforcer rate with R_1 (Equation 29):

$$T_1 = F_{enh}(R_1) \cdot F_{red}(R_2) + T_a \text{ and} \quad (28)$$

$$T_2 = F_{enh}(R_2) \cdot F_{red}(R_1) + T_a. \quad (29)$$

Equations 28 and 29 are the general form of the model for residence time in two-alternative concurrent schedules. It is easy to see why this model will not produce constant values of the generalized matching sensitivity parameter with different overall reinforcer rates. As both enhancing and reducing reinforcer rates increase with increasing overall reinforcer rates, the resulting 3-D surface changes and corresponding values of sensitivity also change (lower-right panel).

The Selection of the Component-Functions Model of Residence Time for conc VI VI Schedules

I assessed 35 molar models of residence time by the values of AICc and BIC. Each model was a product of enhancing- and reducing-component functions plus a constant T_a (Equation 25, or Equations 28 and 29). The models were constructed from all possible combinations of seven enhancing-component functions and five reducing-component functions, giving 35 combinations in total (see Appendix B for the equations of the functions). Reinforcer rate for the first alternative was scaled with a reinforcer rate bias b (bR_1), while reinforcer rate for the second alternative remained intact (R_2). As in the hypothetical example above, the reinforcer rate on the selected alternative was identified with the enhancing reinforcer rate, and the reinforcer rate on the other alternative was identified with the reducing reinforcer rate.

Equations for the component functions were chosen from the equations typically used for single schedules of reinforcement. The set of enhancing-component functions (see Appendix B, Equations B1 to B7) consisted of power functions with two or three adjustable parameters (*df*, degree of freedom) as in Stevens's (1957) law; hyperbolic functions with $df = 2$ and $df = 3$ as in Herrnstein's (1970) model; bounded exponential functions with $df = 2$ and $df = 3$ and a logistic exponential function with $df = 3$ as in Hull's models (Hull, 1943, 1951, 1952; Spence, 1942). All enhancing-component functions are increasing functions with initial values (i.e., the values when the reinforcer rate is zero). All, except power functions, are bounded functions (i.e., they have limiting values when the reinforcer rate is infinity). The power functions were included in the list as they rise to infinity which absolute residence time can also do. Logarithmic functions were excluded to avoid problems with negative values for arguments smaller than 1.

The set of five reducing-component functions (see Appendix B, Equations B8 to B12) consisted of hyperbolic functions with $df = 2$ and $df = 1$, exponential functions with $df = 2$ and $df = 1$, and a logistic function with $df = 2$. The functions were derived from the respective enhancing-component functions. The increasing exponential, hyperbolic, or logistic functions have their initial values smaller than the limiting values. To transform an exponen-tial, hyperbolic, or logistic function into a decreasing one, it is enough to choose the initial values of a function to be greater than their limiting values. Additionally, the initial values for the reducing-component functions were set to 1, so the reducing-component function would not affect the value of residence time in the absence of the reducing reinforcers.

The number of adjustable parameters in the set of 35 models ranged from 5 to 7 degrees of freedom, where 2 degrees of freedom were provided by the reinforcer bias b and residence time constant T_a.

Values of the AICc and BIC, and the means of the variance accounted for (VAC) of the 12 fits (6 pigeons $\times$ 2 schedules) for the three best and three worst models ordered by AICc values are presented in Table 3 (see the complete set of 35 models in Appendix B, Table B1). All models in the set of 35 component-functions models were better models than the Davison and Hunter (1976) Model 2 (compare Tables 1 and 3). The difference between values of AICc and BIC for the Davison and Hunter Model 2 were 143 and 67, respectively – well above the critical value of $\Delta_i = 6$. The VAC for the worst model was 71% against 64% for the Davison and Hunter Model 2.

Both the AICc and BIC information criteria were smallest for the model with the power function ($df = 3$) as the enhancing-component function and with the hyperbolic function ($df = 1$) as the reducing-component function. I will refer to this function as the best component-function model. The *df* for the best model was 6. The AICc and BIC differences from the next best model were significant ($\Delta_i = 16$) with relative likelihood $RL_i = .0004$ (Equation 20) that the model ranked second was better than the model ranked first. The best model had a mean value of VAC = 87% against 64% for the Davison and Hunter (1976) Model 2.

The complete formulation of the best component-functions model is:

$$T_1 = [F_{enh0} + A(bR_1)^{k_{enh}}] \cdot [1 - R_2/(1/k_{red} + R_2)] + T_a \text{ and} \tag{30}$$

$$T_2 = [F_{enh0} + AR_2^{k_{enh}}] \cdot [1 - bR_1/(1/k_{red} + bR_1)] + T_a, \tag{31}$$

Table 3

Accuracy of the three best and three worst component-functions models of residence time (Equations 28 and 29) from all possible combinations of seven enhancing-component functions (see Appendix B, Equations B1 to B7) and five reducing-component functions (see Appendix B, Equations B8 to B12). *Power, Hyp, Exp,* and *Log* are power, hyperbolic, exponential, and logistic component functions. *df* in Columns 2 and 4 are for the component functions; *df* in Column 5 are for the whole model. Other abbreviations are as in Appendix A, Table A1. Models are ordered by values of AICc and ranked separately by AICc and BIC. The data are from Alsop and Elliffe (1988) and Elliffe and Alsop (1996).

Component-functions					AICc			BIC			VAC	
Enhancing	df	Reducing	df	df	Value	Δ_i	Rank	Value	Δ_i	Rank	Mean	SEM
Three best component-functions models as ranked by AICc												
Power	3	Hyp	1	6	−3158	0	1	−2844	0	1	87	1
Exp	3	Hyp	1	6	−3142	16	2	−2828	16	2	86	1
Hyp	3	Hyp	1	6	−3140	18	3	−2825	18	3	86	1
Three worst component-functions models as ranked by AICc												
Hyp	2	Exp	1	5	−2599	560	33	−2334	510	30	72	2
Exp	2	Exp	2	6	−2598	561	34	−2283	561	35	72	3
Exp	2	Exp	1	5	−2573	586	35	−2308	536	34	71	3

where T_1 and T_2 are residence time on the first and second alternatives expressed in s, F_{enh0} is the initial value of the enhancing function in s, A is a scaling coefficient of the enhancing-component function, R_1 and R_2 are obtained reinforcer rates on the first and second alternatives, k_{enh} and k_{red} are enhancing and reducing reinforcer constants, T_a is the residence time constant in s, and b is a reinforcer bias. k_{enh} and k_{red} do not have any dimension, but require R_1 and R_2 to be expressed in reinforcers per min for consistency.

Examples of residence-time fits for the best component-functions model and for the Davison and Hunter (1976) Model 2 are presented in Figure 2. A typical 3-D surface of residence time as a function of enhancing and reducing reinforcer rates is shown in Figure 3, with residence time for the arithmetic schedules being smaller than for the exponential schedules. Accuracy of generalized matching sensitivity predictions was obtained from the sensitivity values calculated for each programmed overall reinforcer rate (for both the arithmetic and exponential schedules with 6 pigeons for each schedule). Seventy-two values of the sensitivity parameter obtained from the data were compared with theoretical values of residence time. The best model accounted for 67% of the variation in generalized matching sensitivity parameter values, whereas the Davison and Hunter Model 2 accounted for −14% (see Figure 4).

The average parameter values for the best model are shown in Table 4, and parameter values for each of 12 fits are given in Appendix B (Table B2). In both arithmetic and exponential groups, there was an outlier that significantly distorted (increased) the mean values of parameters, thus the median value is given to provide a better representation. The parameters k_{enh} and T_a of the exponential-progression schedules were significantly different from those for the arithmetic schedules ($p < .05$ and $< .01$, respectively).

In order to assess any systematic deviation of residence time from the model, the log residence time residuals were approximated by the third-degree polynomial and compared with the same analysis performed for the Davison and Hunter (1976) Model 2. Two different polynomial models were calculated: for the residuals as a function of overall reinforcer rate and for the residuals as a function of relative reinforcer rate. In the case of overall reinforcer rate, each overall rate was used twice (for the residuals of T_1 and T_2). In the case of relative reinforcer rate, the values of $R_1/(R_1 + R_2)$ and $R_2/(R_1 + R_2)$ were used for the residuals of T_1 and T_2, respectively. The Davison and Hunter Model 2 had significant ($p < .05$) systematic deviations in 7 out of 12 overall reinforcer rate fits and in all 12 polynomials for relative residence time. For the component-functions model, there were no cases of systematic deviations. The third-degree polynomials described on average 14% and 35%
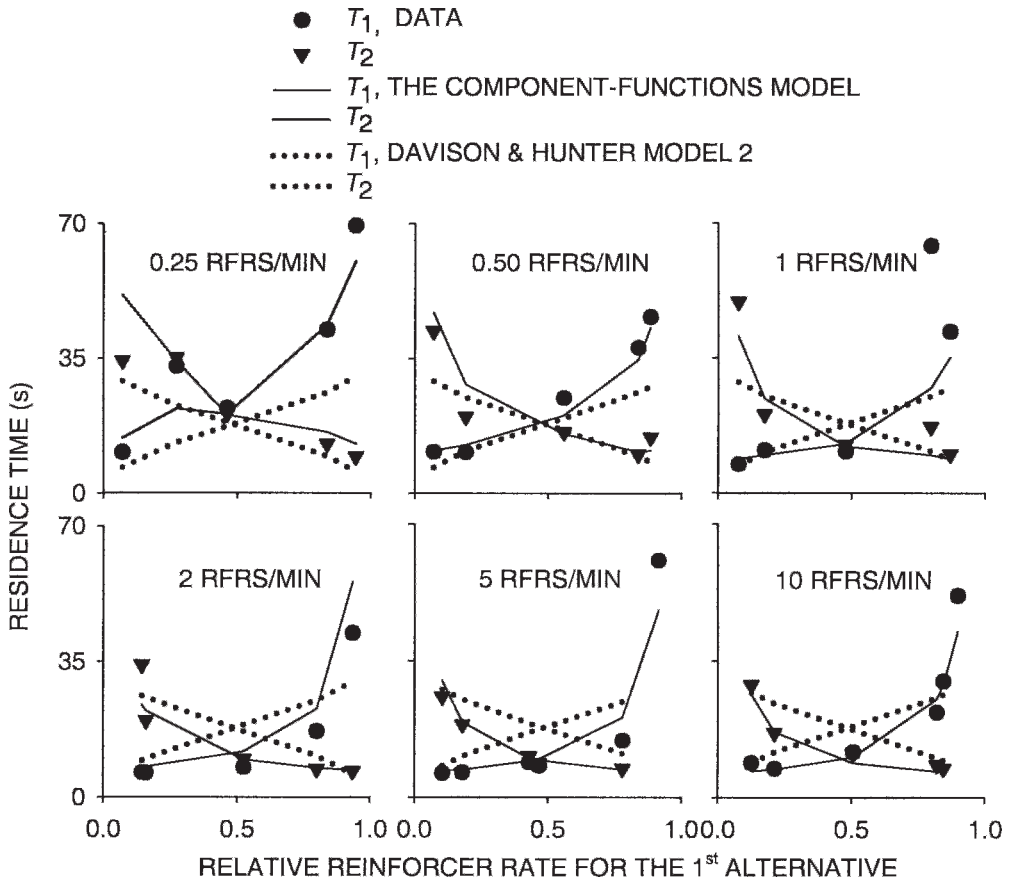
Fig. 2. Predictions of the component-functions model for interdependent conc VI VI performance (solid lines; Equations 30 and 31) versus the Davison and Hunter (1976) Model 2 predictions (dashed lines; Equations 21 and 22). The data are from Pigeon 132 and the exponential schedule type (Alsop & Elliffe, 1996). The parameter values are given in Appendix A, Table A1 and Appendix B, Table B2.
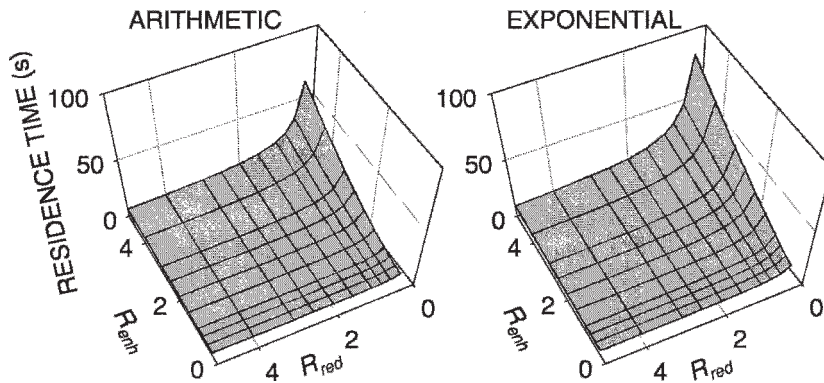
Fig. 3. Models of residence time for arithmetic and exponential schedules. Models included obtained reinforcer rates from 0.1 to 5 reinforcers per min. (Equations 30 and 31; the parameter values appear in Appendix B, Table B2). The data are from Pigeon 132 (Alsop & Elliffe, 1988; Elliffe & Alsop, 1996).
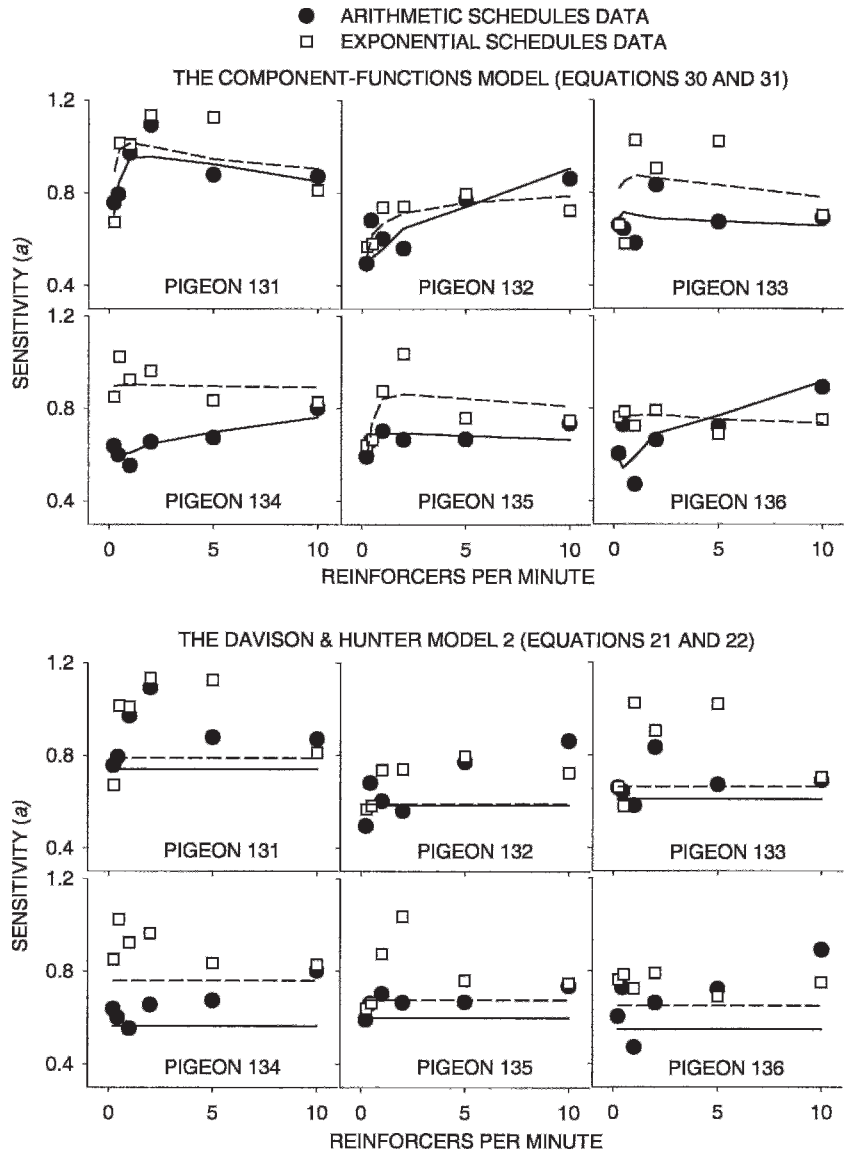
Fig. 4. Generalized matching sensitivity parameter as predicted by two models. The solid lines represent arithmetic schedules and the dashed lines exponential schedules. Parameter values are given in Appendix A, Table A1 and Appendix B, Table B2. Empirical data sensitivities were calculated for each programmed reinforcer rate from residence-time data. The data are from Alsop and Elliffe (1988) and Elliffe and Alsop (1996).

of the variation of residuals in Davison and Hunter Model 2 (overall and relative reinforcer rate polynomials, respectively), whereas for the component-functions model the same values were 4% and 2%, respectively.

DISCUSSION

All component-functions models in the set of 35 that were tested proved to be superior to existing molar models of choice in interdependent conc VI VI schedules and, in particular, to the model proposed by Davison and Hunter (1976), according to both Akaike and Bayesian information criteria. The component-functions model of choice behavior (Equations 30 and 31), with only 6 degrees of freedom, successfully described the results of Alsop and Elliffe's (1988) and Elliffe and Alsop's (1996) experiments with VAC = 87%

Table 4

Parameter values for the best component-functions model (Equations 30 and 31). There are six fits of the models for arithmetic and exponential schedules with 58 to 64 data points in each fit. F_{enh0} and A are initial and scaling values of the enhancing function, k_{enh} and k_{red} are enhancing and reducing reinforcer constants, T_a is the residence time constant, and b is a reinforcer bias. VAC is percentage of variance accounted for.

	Model parameters						
	F_{enh0}	A	k_{red}	k_{enh}	T_a	$log\ b$	VAC
	Arithmetic schedule (Alsop & Elliffe, 1988)						
Mean ± SEM Median	21505	158145	117953	1.143	4.18	−0.08	84
	21452	157982	117865	0.161	0.32	0.02	1
	49	134	57	1.114	4.12	−0.08	84
	Exponential schedule (Elliffe & Alsop, 1996)						
Mean ± SEM Median	23	1732	375	0.861^+	6.44^{++}	−0.11	90
	11	1445	329	0.089	0.31	0.06	1
	12	305	51	0.823	6.12	−0.10	90

[+] $p < .05$ for the comparison of means by 2-tailed paired t-test; [++] $p < .01$.

against VAC = 64% for the Davison and Hunter Model 2 and VAC = 42% for Herrnstein's (1970) hyperbola (Table 2 and 4). Although accurate fitting of data alone is not important per se, failure to fit data accurately by a competing model may be considered to disqualify whatever apparently advantageous theoretical justification the competing model has.

The component-functions model predicted changes in residence time over a wide range of programmed overall reinforcer rates (0.22 to 10 reinforcers per min). By virtue of accurately describing residence times, it also described changes in the generalized matching sensitivity parameter and can describe undermatching, matching, and overmatching as part of the 3-D depiction of the performance of each subject.

Behavioral contrast is an increase in residence time or response rate on a constant reinforcer-rate alternative with decreases in the reinforcer rate on the other alternative and is a feature of conc VI VI schedule performance (Catania, 1963) and multiple-schedule performance (Reynolds, 1961). The component-functions model can predict behavioral contrast as well (see Figure 5).

The model's parameters can be preliminarily interpreted as follows: F_{enh0} and T_a are related to baseline performance (i.e., behavioral activity in the absence of reinforcers); k_{enh} and k_{red} are analogies for the sensitivities to the enhancing and reducing reinforcer rates; and A is a scaling factor for the enhancing-

component function and is to be viewed in conjunction with k_{enh}. To what degree the parameter values are environment-, schedule-, or subject-related is difficult to establish at this point. Additional research will be required.

It seems, however, that the parameter values for the models may not necessarily remain the same across different types of schedules. There were statistically significant differences between the parameter values (k_{enh} and T_a) of the component-functions models for performance in arithmetic and exponential sched-
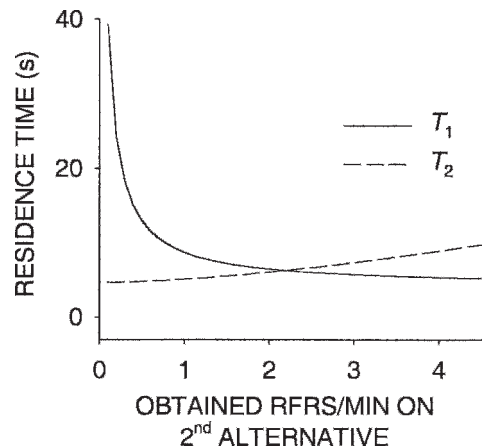

Fig. 5. Behavioral contrast (changes in T_1 as a function of R_2) predicted by the component-functions model for interdependent conc VI VI schedules. The obtained reinforcer rate on the first alternative was two reinforcers per min throughout. Pigeon 133's parameter values are given in Appendix B, Table B2 (Alsop & Elliffe, 1988—arithmetic schedules).

ules (see Table 4). The smaller values of T_a for arithmetic schedules probably are related to the fact that residence time for exponential schedules typically is larger than for arithmetic schedules. Alsop and Elliffe, (1988) and Elliffe and Alsop (1996) reported that changeover rates in exponential schedules were approximately twice those found in arithmetic schedules. The same argument relates to the almost twice larger values of k in Herrnstein's (1970) model and in k' in Davison and Hunter's (1976) Model 2 for performance in exponential schedules compared to performance in arithmetic schedules (Table 2). The variability in k values is well supported (see e.g., Williams, 1988). It seems that the change to a different schedule progression may result in a change in the parameters of the model. It is conceivable that the type of component functions also may change, with only the component-functions-plus-constant structure of the model being invariant (Nevin, 1984).

The model described above was derived from the following starting point: I was looking for a symmetrical way to represent two categories of reinforcers—reinforcers that enhance behavior and those that reduce it (these were termed *enhancers* and *reducers*). I used a set of simultaneous partial differential equations (Equations 26 and 27). The solution has the general form of the product of two independent component functions plus a constant (see Equations 25 and 32).

$$B = F_{enh}(R_{enh}) \cdot F_{red}(R_{red}) + B_a, \qquad (32)$$

where B is a behavioral variable, B_a is the constant, and F_{enh} and F_{red} are enhancing- and reducing-component functions responsible for increasing and reducing behavior, respectively; R_{enh} and R_{red} are enhancing- and reducing-reinforcer rates. According to these differential equations and their solution (Equation 25 and 32), if the quantity of one of the reinforcers (enhancers or reducers) changes, the resulting behavior will change in proportion to the change of its component function and in proportion to the other component function. For behavioral variables like residence time or response rate, the enhancing-component function is an increasing function, and the reducing function is a decreasing one. However, we can imagine a situation where the reverse will be the case,

for example, in measures of latency. It is worth mentioning that enhancers and reducers may belong to the same category of what might be termed *rewarding reinforcers* if they act alone. Yet, when presented to a subject simultaneously in a conc VI VI schedule or a multiple schedule, they acquire the opposing characteristics of enhancers and reducers, driving response competition.

This approach does not assume the maximization or optimization of total reinforcer rate as a primary process (c.f. Houston & McNamara, 1981; Rachlin, 1978, 1982; Rachlin & Burkhard, 1978; Rachlin, Green, Kagel, & Battalio, 1976; Staddon & Motheral, 1978, 1979). In agreement with Prelec (1982) and Davison (1990), in concurrent VI VI schedules the maximization of total obtained reinforcer rate is hard to achieve, since total reinforcer rate changes to a very small degree with changes in relative reinforcer rate. At the same time, the reinforcer rates on alternatives (and consequently, enhancing and reducing reinforcer rates) change dramatically, and so are able to serve as inputs for changes in choice behavior in a law-of-effect type of model. The behavior assumed by the component-functions model is relatively adaptive, inasmuch as it is adaptive for response rate (or residence time) to increase monotonically with an increase in the enhancing reinforcer rate and to decrease monotonically with an increase in the reducing reinforcer rate.

The notion of two tendencies affecting behavior is not new. It can be traced to Pavlovian excitation and inhibition (Pavlov, 1928). Herrnstein's (1970) notion of extraneous reinforcers was intended to play a similar role to that of the reducing reinforcers in the component-functions model. An analogous approach can be seen in the stay- and shift-reinforcers discussed by MacDonall (1999, 2000). Strong parallels to this approach can be found in Staddon's (1977, 1978) modeling of response competition with a postulated ceiling on response rate and an inhibitory effect of the competing responses. A further similarity in approach is that adopted by Dragoi and Staddon (1999) in their acquisition-extinction theory. A multiplicative-components-plus-constant structure was used by Hull (e.g., Equation 35, Hull, 1943, p. 255) to describe the interaction of habit and drive strengths.

The initial value of the reducing function was set to 1 so that, when there are no other reinforcers, the model for two component functions reduces to a model for single-schedule responding:

$$B = F_{enh}(R_{enh}) + B_a. \qquad (33)$$

It may be reasonable to ask why an alternative to Herrnstein's (1970) hyperbola (Equation 3) would be proposed for single-schedule performance. Yet, (a) as already stated, the type of component function can depend on schedule type; (b) Herrnstein's hyperbola for single-schedule performance has not been compared directly with possible alternatives, because the hyperbola was developed as a reduction of the matching model (Equation 1); and (c) the performance of Equation 1 for conc VI VI schedules was inferior to practically all models, except the Stevens's (1957) law-based model.

The component-functions model also extends to multiple-schedule performance, following the example of Herrnstein (1970), and predicts behavioral contrast (see Figure 5). It does not require the introduction of a special parameter (Herrnstein's m) to weaken the interaction of the participating schedules. Further studies need to be conducted to assess the implication of behavioral momentum effects (Nevin, 1974; Nevin & Grace, 2000; Nevin, Mandell & Atak, 1983) and to assess the component-functions model for multiple-schedule performance.

There are no major difficulties in extending the model to choice behavior in concurrent schedules with more than two alternatives, for example, a conc VI VI VI schedule (Equations 34 to 36):

$$B_1 = F_{enh}(R_1) \cdot F_{red}(R_2 + R_3) + B_a, \qquad (34)$$

$$B_2 = F_{enh}(R_2) \cdot F_{red}(R_1 + R_3) + B_a, \qquad (35)$$

$$B_3 = F_{enh}(R_3) \cdot F_{red}(R_1 + R_2) + B_a, \qquad (36)$$

where B_1, B_2, and B_3 are response rates on the first, second, and third alternative of a conc VI VI VI schedule, and R_1, R_2, and R_3 are reinforcer rates on the first, second, and third alternative. The reinforcer rate on the selected alternative is identified with the enhancing reinforcer rate, and the sum of reinforcer rates on the other alternatives is identified with the reducing reinforcer rate. Under other circumstances, an additional parameter—aggregated other reinforcers, R_e—can be added, most likely to the reducing reinforcer rate for the model for a three-alternative concurrent VI VI VI schedule, as well as for the model for a two-alternative conc VI VI schedule. If a generalized matching analysis is performed for a pair of alternatives in such a three-alternative conc VI VI VI schedule, we may expect that a variety of changes in the sensitivity parameter could occur. If, for example, the reinforcer rate on the third alternative is increased, there may be a tendency for the generalized matching sensitivity parameter value for B_1 versus B_2 to decrease. The behavior on Alternatives 1 and 2 in Equations 34 and 35 will be dominated by the increasing value of R_3, affecting the reducing-component function, so that B_1 and B_2 will converge, and sensitivity will decrease. There are some indications that this can be the case (Davison & McCarthy, 1994; Murrell, 1995). However, the changes in sensitivity can take a different shape (see Figure 4), and there can be areas of almost linear log behavior-ratio versus log reinforcer-ratio relations (Davison & Hunter, 1976; Miller & Loveland, 1974; Pliskoff & Brown, 1976). Thus, the model certainly would predict departures from the choice axiom (Luce, 1959), in full agreement with Luce's (1977) later position. In this way, the molar dynamical model to account for the melioration experiments (Herrnstein, 1982; Herrnstein & Vaughan, 1980; Vaughan, 1981) or Mazur's (1992) experiments can be seen as similar to Hull's model for learning (Hull, 1943, Equation 1, p. 119; Spence, 1942).

Herrnstein's (1970) hyperbola (Equation 3) is viewed as a quantitative or relative law of effect (Lattal, 1998; Nevin & Grace, 2000). The model of choice behavior developed here has an adaptive underlying principle based on the symmetrical treatment of enhancers and reducers. The model describes performance in interdependent conc VI VI schedules (Alsop & Elliffe, 1988; Elliffe & Alsop, 1996) with a degree of accuracy higher than existing models and can explain the existence of matching, undermatching, and overmatching in the same subject. Potentially, it can be extended to concurrent schedules with more than two alternatives and to multiple schedules and can be reduced to performance on single-response schedules,

just as Herrnstein's model has been extended. It also can be used to describe the dynamics of behavior between stable states. Thus, it can be considered as an alternative framework for a quantitative law of effect.

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APPENDIX A

ASSESSMENT OF THE EXISTING MOLAR MODELS OF
RESIDENCE TIME

*Set of Models of Residence Time Based on Existing
Molar Approaches*

In the equations below, T_1 and T_2 are residence times on the first and second alternatives, R_1, R_2, and R_0 are obtained reinforcer rates on the first, second, and aggregated other alternatives, c, a, k, and k' are constants, and b is a reinforcer bias.

Stevens's (1957) law (Equation 9)-based model:

$$T_1 = c(bR_1)^a \text{ and} \qquad (A1)$$

$$T_2 = cR_2^a. \qquad (A2)$$

Herrnstein's (1970) hyperbola (Equation 4)-based model:

$$T_1 = kbR_1/(bR_1 + R_2 + R_0) \text{ and} \qquad (A3)$$

$$T_2 = kR_2/(bR_1 + R_2 + R_0). \qquad (A4)$$

Davison and Hunter (1976) Model 1:

$$T_1 = k'[bR_1/(bR_1 + R_2 + R_0)]^a \text{ and} \qquad (A5)$$

$$T_2 = k'[R_2/(bR_1 + R_2 + R_0)]^a. \qquad (A6)$$

Davison and Hunter Model 2:

$$T_1 = k'(bR_1)^a/[(bR_1)^a + R_2^a + R_0^a] \text{ and} \qquad (A7)$$

$$T_2 = k'R_2^a/[(bR_1)^a + R_2^a + R_0^a]. \qquad (A8)$$

McDowell (1986) model (adapted for residence time):

$$T_1 = k'\left[\frac{1}{b_{10}}\left(\frac{R_0}{R_1}\right)^{a_{10}} + \frac{1}{b_{12}}\left(\frac{R_2}{R_1}\right)^{a_{12}} + 1\right]^{-1} \qquad (A9)$$

and

$$T_2 = k'\left[\frac{1}{b_{20}}\left(\frac{R_e}{R_2}\right)^{a_{20}} + b_{12}\left(\frac{R_1}{R_2}\right)^{a_{12}} + 1\right]^{-1}. \qquad (A10)$$

Table A1

Parameter values for the Herrnstein (1970) hyperbola-based model (Equations 23 and 24 or Appendix A, Equations A3 and A4) and Davison and Hunter (1976) Model 2 (Equations 21 and 22 or Appendix A, Equations A7 and A8). The number of data points in each of the 12 fits is between 58 and 64. k and k' are ''the total amount of behavior'', R_0 is the aggregated other reinforcer rate, a is an analog to the generalized matching sensitivity parameter, and b is a reinforcer bias. VAC is percentage of variance accounted for.

		Herrnstein's hyperbolic model				Davison & Hunter Model 2				
		Model parameters				Model parameters				
		k	R_o	log b	VAC	k'	R_o	a	log b	VAC
Arithmetic schedule (Alsop & Elliffe, 1988)										
Pigeon	131	35.2	0	−0.11	48	31.3	0	0.74	−0.12	55
	132	24.2	0	0.00	30	20.4	0	0.59	0.02	64
	133	28.5	0	−0.05	35	24.2	0	0.61	−0.11	59
	134	20.1	0	−0.03	26	16.8	0	0.56	−0.07	66
	135	27.0	0	−0.03	32	23.0	0	0.60	−0.07	58
	136	28.6	0	−0.01	19	23.5	0	0.55	−0.09	55
Mean ± SEM		27.3	0	−0.04	32	23.2	0	0.61	−0.07	60
		2.1	0	0.02	4	2.0	0	0.03	0.02	2
Exponential schedule (Elliffe & Alsop, 1996)										
Pigeon	131	68.2	0	−0.04	59	61.4	0	0.79	−0.08	64
	132	43.2	0	−0.01	30	35.7	0	0.59	0.03	60
	133	42.4	0	−0.15	49	37.6	0	0.66	−0.28	60
	134	35.0	0	−0.01	65	31.5	0	0.76	−0.03	72
	135	47.3	0	−0.04	58	40.7	0	0.68	−0.05	77
	136	44.1	0	0.00	53	38.0	0	0.65	0.01	76
Mean ± SEM		46.7	0	−0.04	52	40.8	0	0.69	−0.07	68
		4.6	0	0.02	5	4.3	0	0.03	0.05	3

APPENDIX B

ASSESSMENT OF THE COMPONENT-FUNCTIONS MODELS OF RESIDENCE TIME

Set of Component-Functions Models of Residence Time

I assessed 35 molar models of residence time. Each model was a product of enhancing- and reducing-component functions plus a constant T_a.

There were seven enhancing-component functions and five reducing-component functions, giving 35 combinations in total. In the description of the component functions, the following abbreviations were used: F is a component function; R is a reinforcer rate, and a, c, d, and k are constants, which are not always consistent between equations and within equations in the body of the article. Constants c and d are the initial and limiting values of a function (i.e., the values when respective reinforcer rates are zero and infinity). The component functions are presented with reinforcer rate in the unbiased form.

The following seven enhancing-component functions with different numbers of adjustable parameters (*df*, degree of freedom) were used:

Power functions with $df = 3$ (Equation B1) and $df = 2$ (Equation B2), similar to Stevens's (1957) law:

$$F = c + aR^k, \qquad (B1)$$

$$F = aR^k. \qquad (B2)$$

Hyperbolic functions with $df = 3$ (Equation B3) and $df = 2$ (Equation B4), similar to Herrnstein's (1970) model:

$$F = c + [(d - c)R]/(1/k + R), \qquad (B3)$$

$$F = c + [dR]/(1/k + R). \qquad (B4)$$

Exponential functions with $df = 3$ (Equation B5) and $df = 2$ (Equation B6), and a logistic exponential function with $df = 3$ (Equation B7), which were extensively used by Hull (1943, 1951, 1952):

$$F = (c - d) \cdot \exp(-kR) + d, \qquad (B5)$$

$$F = d(1 - \exp(-kR)), \qquad (B6)$$

$$F = d/[1 + (d/c - 1) \cdot \exp(-kR)]. \qquad (B7)$$

There were also five reducing component functions, transformed from Equations B3 to B7. The increasing hyperbolic, exponential, or logistic functions have their initial values smaller that the limiting values ($c < d$). To transform a hyperbolic, exponential, or logistic function into a decreasing one, it is enough to choose the initial values of the function to be greater than their limiting values ($c > d$). The initial values for the reducing-component functions were set to 1, which decreased their degrees of freedom by 1. The following functions were used:

Hyperbolic functions with $df = 2$ (Equation B8) and $df = 1$ (Equation B9):

$$F = 1 + [(d - 1)R]/[(1/k + R)] \text{ and} \qquad (B8)$$

$$F = 1 - R/(1/k + R), \qquad (B9)$$

Exponential functions with $df = 2$ (Equation B10) and $df = 1$ (Equation B11) and a logistic exponential function with $df = 2$ (Equation B12):

$$F = (1 - d) \cdot \exp(-kR) + d, \qquad (B10)$$

$$F = \exp(-kR) \text{ and} \qquad (B11)$$

$$F = d/[1 + (d - 1) \cdot \exp(-kR)]. \qquad (B12)$$

Table B1

Accuracy of 35 component-functions models of residence time (Equations 28 and 29) as all possible combinations of seven enhancing-component functions (see Appendix B, Equations B1 to B7) and five reducing-component functions (see Appendix B, Equations B8 to B12). *Power, Hyp, Exp*, and *Log* are power, hyperbolic, exponential, and logistic component functions. *df* in Columns 2 and 4 are for component functions; *df* in Column 5 are for the whole model. Other abbreviations are as in Appendix A, Table A1. Models are ordered by values of AICc. The data were from Alsop and Elliffe (1988) and Elliffe and Alsop (1996).

Component-functions					AICc			BIC			VAC	
Enhancing	*df*	Reducing	*df*	*df*	Value	Δ_i	Rank	Value	Δ_i	Rank	Mean	SEM
Power	3	Hyp	1	6	−3158	0	1	−2844	0	1	87	1
Exp	3	Hyp	1	6	−3142	16	2	−2828	16	2	86	1
Hyp	3	Hyp	1	6	−3140	18	3	−2825	18	3	86	1
Power	3	Hyp	2	7	−3138	20	4	−2775	69	5	87	1
Exp	3	Hyp	2	7	−3133	26	5	−2769	74	6	87	1
Hyp	3	Hyp	2	7	−3132	26	6	−2769	75	7	87	1
Hyp	3	Log	2	7	−3128	30	7	−2764	79	8	87	1
Power	3	Log	2	7	−3118	40	8	−2754	89	9	87	1
Exp	3	Log	2	7	−3091	68	9	−2727	117	12	86	1
Power	2	Hyp	1	5	−3076	83	10	−2811	32	4	85	2
Power	2	Hyp	2	6	−3061	97	11	−2746	97	10	85	2
Log	3	Hyp	2	7	−3048	110	12	−2685	159	15	86	1
Log	3	Hyp	1	6	−3047	112	13	−2732	112	11	85	1
Power	2	Log	2	6	−3027	131	14	−2713	131	13	84	2
Hyp	2	Hyp	1	5	−2959	199	15	−2695	149	14	82	2
Exp	2	Hyp	1	5	−2940	218	16	−2676	168	16	81	2
Hyp	2	Hyp	2	6	−2932	226	17	−2618	226	17	82	2
Exp	2	Hyp	2	6	−2914	245	18	−2599	245	18	81	2
Power	3	Exp	2	7	−2898	260	19	−2535	309	19	83	1
Log	3	Log	2	7	−2866	292	20	−2503	341	22	82	1
Hyp	3	Exp	2	7	−2863	295	21	−2500	344	23	82	2
Exp	3	Exp	2	7	−2862	296	22	−2499	345	24	82	2
Log	3	Exp	2	7	−2851	307	23	−2488	356	25	82	1
Hyp	2	Log	2	6	−2830	328	24	−2515	328	20	79	3
Power	2	Exp	2	6	−2825	333	25	−2511	333	21	80	2
Exp	2	Log	2	6	−2750	409	26	−2435	409	26	76	3
Hyp	2	Exp	2	6	−2694	464	27	−2379	464	28	76	2
Power	3	Exp	1	6	−2666	492	28	−2352	492	29	76	2
Power	2	Exp	1	5	−2657	501	29	−2393	451	27	74	2
Exp	3	Exp	1	6	−2646	512	30	−2332	512	31	75	2
Log	3	Exp	1	6	−2642	516	31	−2328	516	32	75	2
Hyp	3	Exp	1	6	−2633	525	32	−2319	525	33	75	2
Hyp	2	Exp	1	5	−2599	560	33	−2334	510	30	72	2
Exp	2	Exp	2	6	−2598	561	34	−2283	561	35	72	3
Exp	2	Exp	1	5	−2573	586	35	−2308	536	34	71	3

Table B2

Parameter values for the best component-functions model (Equations 30 and 31). The number of data points in each of the 12 fits was between 58 and 64. F_{enh0} and A are initial and scaling values of the enhancing function, k_{enh} and k_{red} are enhancing and reducing reinforcer constants, T_a is the residence time constant, and b is a reinforcer bias. VAC is percentage of variance accounted for.

| | | Model parameters | | | | | | |
		F_{enh0}	A	k_{red}	k_{enh}	T_a	$\log b$	VAC
Arithmetic schedule (Alsop & Elliffe, 1988)								
Pigeon	131	24	113	22	0.58	2.95	−0.09	83
	132	28	32	29	1.56	4.18	0.04	84
	133	65	154	83	0.96	4.49	−0.12	84
	134	128764	948056	707277	1.26	4.06	−0.08	87
	135	33	59	31	0.92	4.03	−0.13	84
	136	113	455	273	1.57	5.38	−0.08	81
Mean ± SEM		21505	158145	117953	1.143	4.18	−0.08	84
Median		21452	157982	117865	0.161	0.32	0.02	1
		49	134	57	1.114	4.12	−0.08	84
Exponential schedule (Elliffe & Alsop, 1996)								
Pigeon	131	19	651	65	0.71	7.26	−0.20	88
	132	61	61	26	1.20	5.88	0.08	90
	133	0	230	37	0.64	5.90	−0.31	84
	134	0	8945	2021	1.00	5.74	−0.15	94
	135	5	122	15	0.68	6.35	−0.01	90
	136	50	380	88	0.94	7.51	−0.06	94
Mean ± SEM		23	1732	375	0.861	6.44	−0.11	90
Median		11	1445	329	0.089	0.31	0.06	1
		12	305	51	0.823	6.12	−0.10	90